(e) A closed cylindrical tank is to have a surface area of 20 m^2.

Find the radius the tank needs to have so that the volume it can hold is as large as possible.

You do not have to prove that your solution gives the maximum volume.

Show any derivatives that you need to find when solving this problem.

ASSESSOR'S USE ONLY

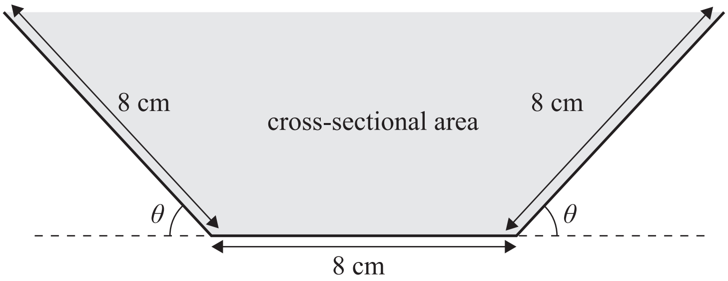
(e)	$20 = 2\pi r^2 + 2\pi rh$ $2\pi r(r+h) = 20$ $h = \frac{10}{\pi r} - r$ $V = \pi r^2 h = \pi r^2 \cdot \left(\frac{10}{\pi r} - r\right)$ $V = 10r - \pi r^3$ $\frac{dV}{dr} = 10 - 3\pi r^2$ $\frac{dV}{dr} = 0 \Rightarrow r = \sqrt{\frac{10}{3\pi}}$ or $r = 1.03$ m OR $20 = \pi r^2 + 2\pi rh$ $V = 10r - \frac{\pi r^3}{2}$ $\frac{dV}{dr} = 10 - \frac{3\pi r^2}{2}$ $r = \sqrt{\frac{20}{3\pi}} = 1.46$		Equation for volume in terms of 1 variable found, and differentiated correctly.	Problem solved including correct derivative.

NØ = No response / no relevant evidence
N1 = ONE question demonstrating limited knowledge of differentiation techniques
N2 = ONE correct derivative
A3 = TWO of Achievement
A4 = THREE of Achievement
M5 = ONE of Merit
M6 = TWO of Merit
E7 = Excellence with minor errors ignored
E8 = Excellence correct

ASSESSOR'S USE ONLY

(e) A copper sheet of width 24 cm is folded, as shown, to make spouting.

Cross-section:



Find angle θ which gives the maximum cross-sectional area.

You do not need to prove that you have found a maximum.

Show any derivatives that you need to find when solving this problem.

Two	Expected Coverage	Achievement	Merit	Excellence
(a)	$\frac{dy}{dx}=\frac{1}{3}\left(\pi-x^2\right)^{\frac{-2}{3}}\cdot -2x$ or $\frac{dy}{dx}=\frac{-2x}{3\left(\pi-x^2\right)^{\frac{2}{3}}}$	Correct derivative.		
(b)	$\frac{dy}{dx}=3\left(x^3-2x\right)^2\cdot\left(3x^2-2\right)$ At $x=1$, $\frac{dy}{dx}=3\cdot\left(-1\right)^2\cdot 1=3$ At $x=1$, $y=-1$ $y+1=3(x-1)$ $y=3x-4$	Correct solution with correct derivative shown.		
(c)	$f'(x)=1-e^x+\frac{k}{x^2}$ $f'(x)=0 \quad \Rightarrow 1-e^{-1}+k=0$ $k=e^{-1}-1$ Or $k=-0.632$	Correct derivative.	Correct value for k and correct derivative.	
(d)(i) (ii) (iii)	1. $x=1$ 2. $x>3$ 3. $-2, -1, 3$ -3 Does not exist.		THREE correct answers (out of 5).	
(e)	$A(\theta)=64\sin\theta+64\sin\theta\cos\theta$ OR $A(\theta)=64\sin\theta+32\sin 2\theta$ $A'(\theta)=64\cos\theta+64\cos^2\theta-64\sin^2\theta$ OR $A'(\theta)=64\cos\theta+64\cos 2\theta$ $=64\cos\theta+64\cos^2\theta-64(1-\cos^2\theta)$ $=64(2\cos^2\theta+\cos\theta-1)$ Minimum when $A'(\theta)=0$ $2\cos^2\theta+\cos\theta-1=0$ $\left(2\cos\theta-1\right)\left(\cos\theta+1\right)=0$ Or $\cos\theta=\frac{1}{2}$ or $\cos\theta=-1$ (NO) $\theta=60°$ or $\theta=\frac{\pi}{3}$		Correct derivative.	Correct solution with correct derivatives.

NØ = No response / no relevant evidence
N1 = ONE question demonstrating limited knowledge of differentiation techniques
N2 = ONE correct derivative
A3 = TWO of Achievement
A4 = THREE of Achievement
M5 = ONE of Merit
M6 = TWO of Merit
E7 = Excellence with minor errors ignored
E8 = Excellence correct

ASSESSOR'S USE ONLY

(e) A spherical balloon is being inflated with helium.

The balloon is being inflated in such a way that its volume is increasing at a constant rate of $300\ \text{cm}^3\ \text{s}^{-1}$.

The material that the balloon is made of is of limited strength, and the balloon will burst when its surface area reaches $7\,500\ \text{cm}^2$.

Find the rate at which the surface area of the balloon is increasing when it reaches bursting point.

Show any derivatives that you need to find when solving this problem.

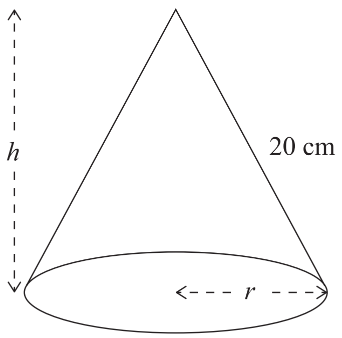
(e)	$\frac{dV}{dt} = 300$ $A = 4\pi r^2 \quad \Rightarrow \frac{dA}{dr} = 8\pi r$ $V = \frac{4}{3}\pi r^3 \quad \Rightarrow \frac{dV}{dr} = 4\pi r^2$ $\frac{dA}{dt} = \frac{dV}{dt} \cdot \frac{dA}{dr} \cdot \frac{dr}{dV}$ $= \frac{2400\pi r}{4\pi r^2}$ $= \frac{600}{r}$ $A = 7500 \quad \Rightarrow 4\pi r^2 = 7500$ $r = \sqrt{\frac{7500}{4\pi}} = 24.43 \text{ cm}$ $\therefore \frac{dA}{dt} = \frac{600}{24.43} = 24.56 \text{ cm}^2\,\text{s}^{-1}$	Correct expressions for $\frac{dV}{dr}$ and $\frac{dA}{dr}$	Correct expressions for $\frac{dV}{dr}$, $\frac{dA}{dr}$ and $\frac{dA}{dt}$	Correct solution along with correct expressions for $\frac{dV}{dr}$, $\frac{dA}{dr}$ and $\frac{dA}{dt}$

NØ = No response / no relevant evidence
N1 = ONE question demonstrating limited knowledge of differentiation techniques
N2 = ONE correct derivative
A3 = TWO of Achievement
A4 = THREE of Achievement
M5 = ONE of Merit
M6 = TWO of Merit
E7 = Excellence with minor errors ignored
E8 = Excellence correct

Judgement Statement

	Not Achieved	Achievement	Achievement with Merit	Achievement with Excellence
Score range	0 – 8	9 – 13	14 – 20	21 – 24

(e) What is the maximum volume of a cone if the slant length of the cone is 20 cm?



You do not need to prove that the volume you have found is a maximum.

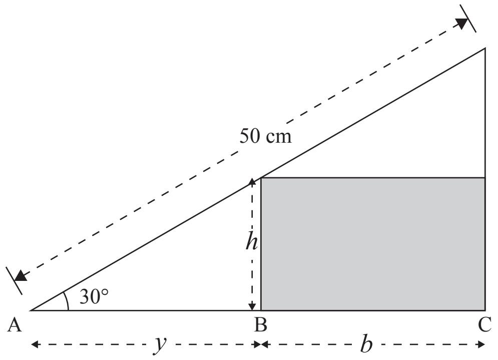
Show any derivatives that you need to find when solving this problem.

ASSESSOR'S USE ONLY

(e)	$h^2 + r^2 = 400$ $h = \sqrt{400 - r^2}$ $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2\sqrt{400 - r^2}$ $\frac{dV}{dr} = \frac{2}{3}\pi r\sqrt{400 - r^2} + \frac{1}{3}\pi r^2 . \frac{1}{2}(400 - r^2)^{\frac{-1}{2}} . -2r$ $\frac{dV}{dr} = \frac{\frac{2}{3}\pi r(400 - r^2) - \frac{1}{3}\pi r^3}{\sqrt{400 - r^2}}$ At maximum volume: $\frac{dV}{dR} = 0$ $2(400 - r^2) = r^2$ $3r^2 = 800$ $r = 16.3\text{ cm}$ $V = 3225\text{ cm}^3$ Alternative working: $r^2 = 400 - h^2$ $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(400 - h^2)h$ $= \frac{\pi}{3}(400h - h^3)$ $\frac{dV}{dh} = \frac{\pi}{3}(400 - 3h^2)$ At maximum, $\frac{dV}{dh} = 0$ $400 - 3h^2 = 0$ $h^2 = \frac{400}{3}$ $h = \frac{20}{\sqrt{3}} = 11.547\text{ cm}$ $V = 3225\text{ cm}^3$	Correct derivative for an incorrect but relevant expression for V.	A correct expression for $\frac{dV}{dr}$.	A correct solution. Units not required.

NØ	N1	N2	A3	A4	M5	M6	E7	E8
No response; no relevant evidence.	ONE answer demonstrating limited knowledge of differentiation techniques.	ONE correct derivative	2u	3u	1r	2r	1t with minor error(s).	1t

(e) A rectangle is drawn inside a right angled triangle, as shown in the diagram below.

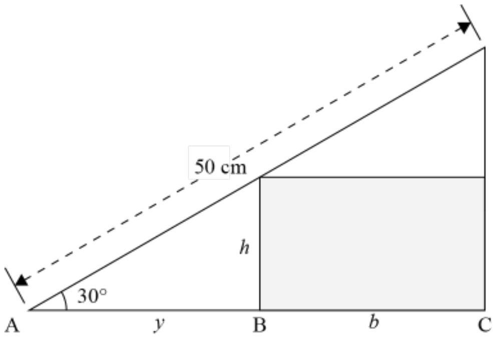


Point B moves along the base of the triangle AC, beginning at point A, at a constant speed of 3 cm s^{-1}.

At what rate is the area of the rectangle changing when point B is 20 cm from point A?

Show any derivatives that you need to find when solving this problem.

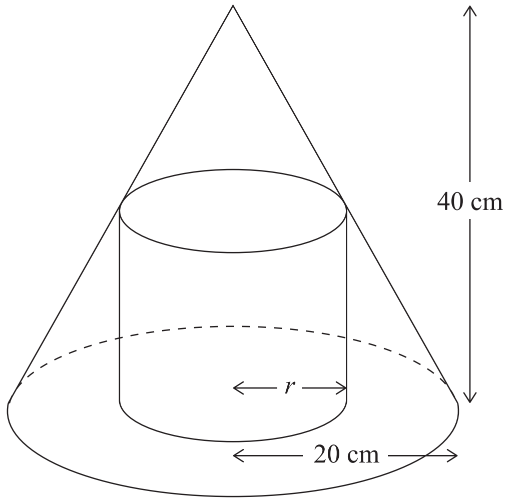
ASSESSOR'S USE ONLY

(e)	 $\tan 30 = \frac{h}{y}$ $h = y\tan 30$ $\cos 30 = \frac{y+b}{50}$ $y + b = 50\cos 30$ $b = 50\cos 30 - y$ Area = base × height $A = (50\cos 30 - y)(y\tan 30)$ $= 50y\sin 30 - y^2\tan 30$ $= 25y - \frac{y^2}{\sqrt{3}}$ $\frac{dA}{dy} = 25 - \frac{2y}{\sqrt{3}}$ At $y = 20$ $\frac{dA}{dy} = 25 - \frac{40}{\sqrt{3}}$ $\frac{dA}{dt} = \frac{dA}{dy} \times \frac{dy}{dt}$ $= \left(25 - \frac{40}{\sqrt{3}}\right) \times 3$ $= 5.72 \text{ cm}^2 \text{ s}^{-1}$	Correct derivative for an incorrect but relevant expression for A.	A correct expression for $\frac{dA}{dy}$	A correct solution. Units not Required.

NØ	N1	N2	A3	A4	M5	M6	E7	E8
No response; no relevant evidence.	ONE answer demonstrating limited knowledge of differentiation techniques.	ONE correct derivative	2u	3u	1r	2r	1t with minor error(s).	1t

(e) A cone has a radius of 20 cm and a height of 40 cm.

A cylinder fits inside the cone, as shown below.



What must the radius of the cylinder be to give the cylinder the maximum volume?

You do not need to prove that the volume you have found is a maximum.

Show any derivatives that you need to find when solving this problem.

ASSESSOR'S USE ONLY

Year 2014 - Question 3(e) - ANSWER

(e)	$h = 40 - 2r$ $V = \pi r^2 h$ $= \pi r^2(40 - 2r)$ $= 40\pi r^2 - 2\pi r^3$ $\frac{dV}{dr} = 80\pi r - 6\pi r^2$ $\frac{dV}{dr} = 0 \Rightarrow 80\pi r - 6\pi r^2 = 0$ $2\pi r(40 - 3r) = 0$ $r = \frac{40}{3}$ or 0 $r = \frac{40}{3}$ cm	Correct derivative for an incorrect but relevant expression for V.	A correct expression for $\frac{dV}{dr}$	A correct solution. Units not required.

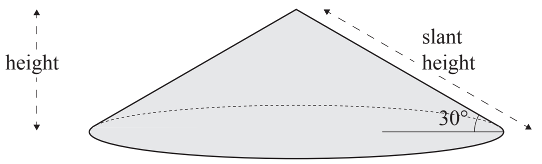
NØ	N1	N2	A3	A4	M5	M6	E7	E8
No response; no relevant evidence.	ONE answer demonstrating limited knowledge of differentiation techniques.	ONE correct derivative	2u	3u	1r	2r	1t with minor error(s).	1t

Cut Scores

	Not Achieved	Achievement	Achievement with Merit	Achievement with Excellence
Score range	0 – 7	8 – 12	13 – 20	21 – 24

ASSESSOR'S USE ONLY

(e) Salt harvested at the Grassmere Saltworks forms a cone as it falls from a conveyor belt. The slant of the cone forms an angle of 30° with the horizontal. The conveyor belt delivers the salt at a rate of 2 m^3 of salt per minute.



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Find the rate at which the slant height is increasing when the radius of the cone is 10 m.

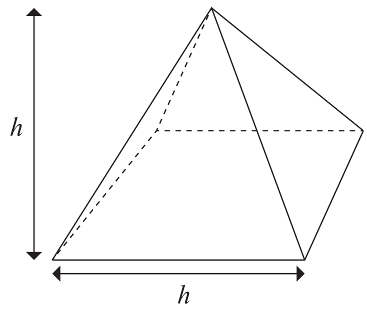
You must use calculus and show any derivatives that you need to find when solving this problem.

(e)	Let V = volume (m^3) S = slant height (m) h = height (m) r = radius (m) $\cos 30 = \frac{r}{S}$ $S = \frac{r}{\cos 30}$ $\frac{dS}{dr} = \frac{1}{\cos 30}$ $\tan 30 = \frac{h}{r}$ $h = r \tan 30$ $V = \frac{1}{3}\pi r^2 h$ $= \frac{1}{3}\pi r^3 \tan 30$ $\frac{dV}{dr} = \pi r^2 \tan 30$ $\frac{dS}{dt} = \frac{dS}{dr} \times \frac{dr}{dV} \times \frac{dV}{dt}$ $= \frac{1}{\cos 30} \times \frac{1}{\pi r^2 \tan 30} \times 2$ When $r = 10$ m, $\frac{dS}{dt} = \frac{1}{\cos 30} \times \frac{1}{\pi 10^2 \times \tan 30} \times 2$ $= 0.01273$ m / minute	$\frac{dS}{dr}$ or $\frac{dV}{dr}$ correct.	Valid statement of the relationship between rates.	Correct solution with correct derivatives.

NØ	N1	N2	A3	A4	M5	M6	E7	E8
No response; no relevant evidence.	ONE answer demonstrating limited knowledge of differentiation techniques.	1u	2u	3u	1r	2r	1t with minor error(s).	1t

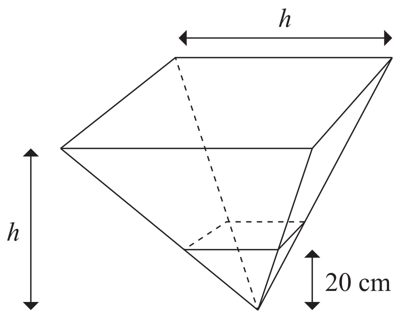
ASSESSOR'S USE ONLY

(e) A water container is constructed in the shape of a square-based pyramid. The height of the pyramid is the same as the length of each side of its base.



A vertical height of 20 cm is then cut off the top of the pyramid, and a new flat top added.

The pyramid is then inverted and water is poured in at a rate of 3000 cm^3 per minute.



Find the rate at which the surface area of the water is increasing when the depth of the water is 15 cm.

$$\text{Volume of pyramid} = \frac{1}{3} \times \text{base area} \times \text{height}$$

You must use calculus and show any derivatives that you need to find when solving this problem.

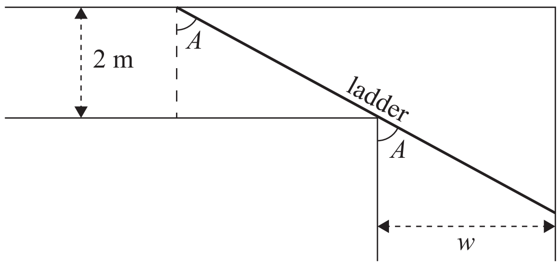
(e)	Depth of water = x $h = x + 20$ $V = \frac{1}{3}h^3 - \frac{1}{3}20^3$ $= \frac{1}{3}(x+20)^3 - \frac{1}{3}20^3$ $\frac{dV}{dx} = (x+20)^2$ $A = (x+20)^2$ $\frac{dA}{dx} = 2(x+20)$ $\frac{dV}{dt} = 3000$ $\frac{dA}{dt} = \frac{dA}{dx} \times \frac{dx}{dV} \times \frac{dV}{dt}$ $= 2(x+20) \times \frac{1}{(x+20)^2} \times 3000$ When $x = 15$ $\frac{dA}{dt} = 2 \times 35 \times \frac{1}{35^2} \times 3000 = 171.4 \text{ cm}^2 \text{ min}^{-1}$	Correct $\frac{dV}{dx}$ OR $\frac{dA}{dx}$	Correct $\frac{dV}{dx}$ AND $\frac{dA}{dx}$	Correct solution.

NØ	N1	N2	A3	A4	M5	M6	E7	E8
No response; no relevant evidence.	ONE answer demonstrating limited knowledge of differentiation techniques.	1u	2u	3u	1r	2r	1t with minor error(s).	1t

ASSESSOR'S USE ONLY

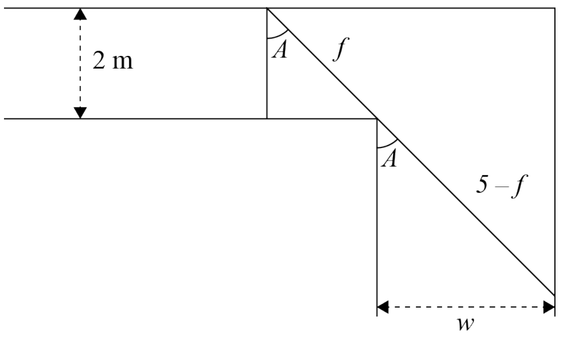
(e) A corridor is 2 m wide.

At the end it turns 90° into another corridor.



What is the minimum width, w, of the second corridor if a ladder of length 5 m can be carried horizontally around the corner?

You must use calculus and show any derivatives that you need to find when solving this problem.

(e)	 $\cos A = \frac{2}{f}$ $f = \frac{2}{\cos A}$ $\sin A = \frac{w}{5-f}$ $w = (5-f)\sin A$ $= \left(5 - \frac{2}{\cos A}\right)\sin A$ $= 5\sin A - 2\tan A$ $\frac{dw}{dA} = 5\cos A - 2\sec^2 A$ $\frac{dw}{dA} = 0 \Rightarrow 5\cos A - 2\sec^2 A = 0$ $5\cos^3 A - 2 = 0$ $\cos^3 A = \frac{2}{5}$ $A = 42.5°$ $w = 1.55$ m	Differentiate corretedly related but incorrect expression for w.	Correct $\frac{dw}{dA}$	Correct solution.

NØ	N1	N2	A3	A4	M5	M6	E7	E8
No response; no relevant evidence.	ONE answer demonstrating limited knowledge of differentiation techniques.	1u	2u	3u	1r	2r	1t with minor error(s).	1t

Cut Scores

Not Achieved	Achievement	Achievement with Merit	Achievement with Excellence
0 – 7	8 – 12	13 – 18	19 – 24

(e) A curve is defined by the function $f(x) = e^{-(x-k)^2}$.

Find, in terms of k, the x-coordinate(s) for which $f''(x) = 0$.

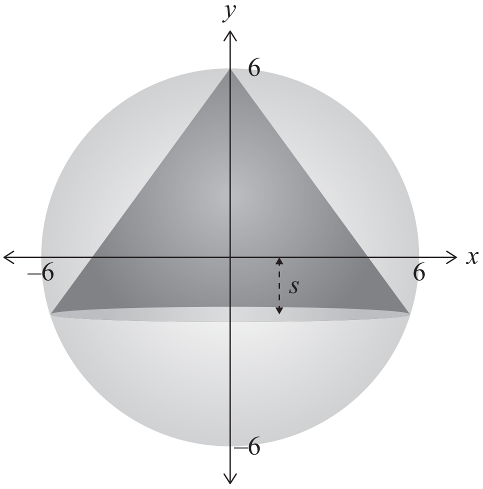
You must use calculus and show any derivatives that you need to find when solving this problem.

(e)	$f(x)=\mathrm{e}^{-(x-k)^2}$ $f'(x)=-2(x-k)\mathrm{e}^{-(x-k)^2}$ $f''(x)=-2\mathrm{e}^{-(x-k)^2}+4(x-k)^2\,\mathrm{e}^{-(x-k)^2}$ $=\mathrm{e}^{-(x-k)^2}\left[4(x-k)^2-2\right]$ $f''(x)=0 \Rightarrow 4(x-k)^2-2=0$ $4(x-k)^2=2$ $(x-k)^2=\frac{1}{2}$ $(x-k)=\frac{\pm 1}{\sqrt{2}}$ $x=k\pm\frac{1}{\sqrt{2}}$	Correct $f'(x)$	Correct $f''(x)$	Correct solutions with correct $f'(x)$ and $f''(x)$

NØ	N1	N2	A3	A4	M5	M6	E7	E8
No response; no relevant evidence.	ONE answer demonstrating limited knowledge of differentiation techniques.	1u	2u	3u	1r	2r	1t with minor error(s).	1t

ASSESSOR'S USE ONLY

(e) A cone of height h and radius r is inscribed, as shown, inside a sphere of radius 6 cm.



The base of the cone is s cm below the x-axis.

Find the value of s which maximises the volume of the cone.

You must use calculus and show any derivatives that you need to find when solving this problem.

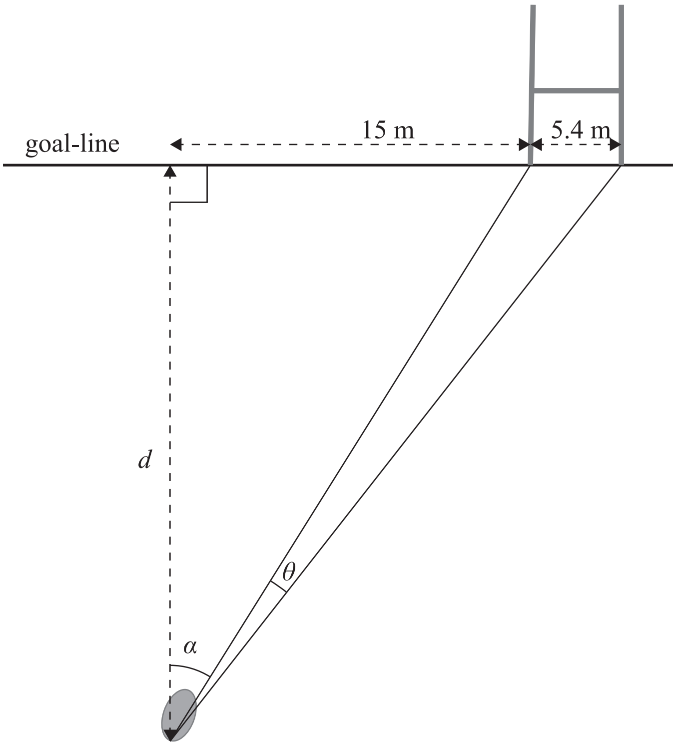
You do not need to prove that the volume you have found is a maximum.

(e)	$Vol = \frac{1}{3}\pi r^2 h$ $h = 6 + s$ $s^2 + r^2 = 6^2$ $r^2 = 36 - s^2$ $\therefore V = \frac{1}{3}\pi(36 - s^2)(6 + s)$ $= \frac{1}{3}\pi(216 + 36s - 6s^2 - s^3)$ $\frac{dV}{ds} = \frac{1}{3}\pi(36 - 12s - 3s^2)$ Max volume when $\frac{dV}{ds} = 0$ $\Rightarrow 3s^2 + 12s - 36 = 0$ $s^2 + 4s - 12 = 0$ $(s+6)(s-2) = 0$ $s = -6$ or $s = 2$ $s = 2$		Correct expression for $\frac{dV}{ds}$	Correct solution.

NØ	N1	N2	A3	A4	M5	M6	E7	E8
No response; no relevant evidence.	ONE answer demonstrating limited knowledge of differentiation techniques.	1u	2u	3u	1r	2r	1t with minor error(s).	1t

(e) In a rugby game, a try is scored 15 m from the left-hand goal-post. The conversion kick is taken at some point on the line perpendicular to the goal-line from the point where the try was scored, as shown in the diagram below.

The ball needs to pass between the goal-posts, which are 5.4 m apart.



Find the distance d from the goal-line that the conversion kick should be taken from in order to maximise the angle θ between the lines from the ball to the goal-posts.

You must use calculus and show any derivatives that you need to find when solving this problem.

You do not need to prove that the angle you have found is a maximum.

ASSESSOR'S USE ONLY

(e)	$\tan\alpha = \frac{15}{d}$ $\qquad \tan(\alpha+\theta) = \frac{20.4}{d}$ $\tan\theta = \tan((\alpha+\theta)-\alpha)$ $= \frac{\tan(\alpha+\theta)-\tan\alpha}{1-\tan(\alpha+\theta)\cdot\tan\alpha}$ $= \frac{\frac{20.4}{d}-\frac{15}{d}}{1+\frac{20.4\times15}{d^2}}$ $= \frac{\frac{5.4}{d}}{\frac{d^2+306}{d^2}}$ $= \frac{5.4d}{d^2+306}$ Max when $\frac{\mathrm{d}(\tan\theta)}{\mathrm{d}d} = 0$ $\frac{(d^2+306)\times5.4-5.4d\times2d}{(d^2+306)^2} = 0$ $5.4d^2+306\times5.4-10.8d^2 = 0$ $5.4d^2-306\times5.4 = 0$ $d^2 = 306$ $d = 17.5$ m		Correct expression for $\frac{\mathrm{d}(\tan\theta)}{\mathrm{d}d}$ or $\frac{\mathrm{d}\theta}{\mathrm{d}d}$	Correct solution – units not required.

NØ	N1	N2	A3	A4	M5	M6	E7	E8
No response; no relevant evidence.	ONE answer demonstrating limited knowledge of differentiation techniques.	1u	2u	3u	1r	2r	1t with minor error(s).	1t

Cut Scores

Not Achieved	Achievement	Achievement with Merit	Achievement with Excellence
0–7	8–12	13–18	19–24

(e) Find the values of a and b such that the curve $y = \frac{ax - b}{x^2 - 1}$ has a turning point at (3,1).

You must use calculus and show any derivatives that you need to find when solving this problem.

ASSESSOR'S USE ONLY

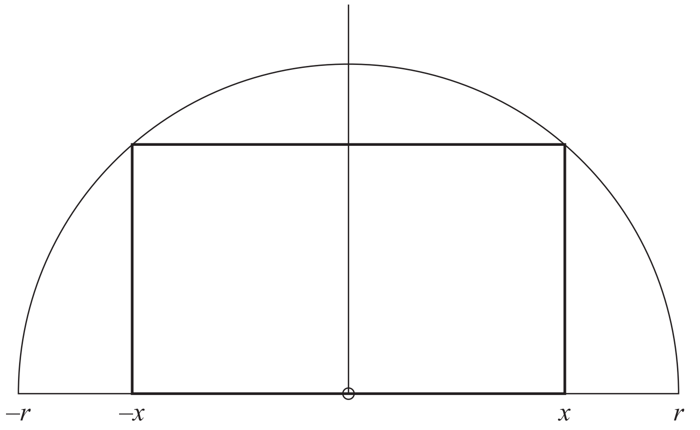
Assessment Schedule – 2017

Calculus: Apply differentiation methods in solving problems (91578)

Evidence Statement

Q 1	Evidence	Achievement	Merit	Excellence
(a)	$\frac{1}{2}x^{\frac{-1}{2}} + 2\sec^2(2x)$	Correct solution.		
(b)	$\frac{dy}{dx} = \frac{(x+2).2e^{2x} - e^{2x}}{(x+2)^2}$ At $x = 0$ $\frac{dy}{dx} = \frac{2\times2-1}{4} = \frac{3}{4}$	Correct solution with correct derivative.		
(c)	$y = 0.5(x-3)^2 + 2$ $\frac{dy}{dx} = 2\times0.5\times(x-3)$ $= x - 3$ At $x = 1$ $\frac{dy}{dx} = -2$ $\therefore$ For normal $\frac{dy}{dx} = \frac{1}{2}$ Through (1, 4) $\therefore$ Eqn of normal $y = \frac{1}{2}x + 3.5$ At point P: $\frac{1}{2}x + 3.5 = 0.5(x-3)^2 + 2$ $x + 7 = (x-3)^2 + 4$ $x + 7 = x^2 - 6x + 9 + 4$ $x^2 - 7x + 6 = 0$ $(x-6)(x-1) = 0$ At point P $x = 6$	Correct expression for $\frac{dy}{dx}$ (i.e. correct derivative).	Correct solution with correct derivative.	
(d)	$\frac{dx}{dt} = \frac{1}{2}(t+1)^{\frac{-1}{2}} = \frac{1}{2\sqrt{t+1}}$ $\frac{dy}{dt} = 2\cos 2t$ $\frac{dy}{dx} = 2\cos 2t.2\sqrt{t+1}$ $= 4\cos 2t.\sqrt{t+1}$ At $t = 0$ $\frac{dy}{dx} = 4\cos 0\times\sqrt{1}$ $= 4$	$\frac{dx}{dt}$ or $\frac{dy}{dt}$ correct.	Correct solution with correct derivatives.	
(e)	$\frac{dy}{dx} = \frac{(x^2-1).a - (ax-b).2x}{(x^2-1)^2}$ At $x = 3$, $\frac{dy}{dx} = 0$ $\Rightarrow 8a - (3a-b)\times6 = 0$ $-10a + 6b = 0$ $5a = 3b$	Correct derivative.	Correct derivative plus one of the two equations relating *a* and *b*.	Correct solution with correct derivative.

(e) A rectangle is inscribed in a semi-circle of radius r, as shown below.



Show that the maximum possible area of such a rectangle occurs when $x = \frac{r}{\sqrt{2}}$.

You do not need to prove that your solution gives the maximum area.

You must use calculus and show any derivatives that you need to find when solving this problem.

	Alternative: $d = \left(x^2 - 7x + 16\right)^{\frac{1}{2}}$ $\frac{\mathrm{d}d}{\mathrm{d}x} = \frac{1}{2}\left(x^2 - 7x + 16\right)^{\frac{-1}{2}}.(2x - 7)$ $= \frac{2x - 7}{2\sqrt{x^2 - 7x + 16}}$ Minimum when $\frac{\mathrm{d}d}{\mathrm{d}x} = 0$ $2x - 7 = 0$ etc			
(e)	$\text{Area} = 2x\sqrt{r^2 - x^2}$ $A(x) = 2x\left(r^2 - x^2\right)^{\frac{1}{2}}$ $A'(x) = 2\left(r^2 - x^2\right)^{\frac{1}{2}} + 2x.\frac{1}{2}\left(r^2 - x^2\right)^{\frac{-1}{2}}.(-2x)$ $= 2\sqrt{r^2 - x^2} - \frac{2x^2}{\sqrt{r^2 - x^2}}$ $A'(x) = 0 \Rightarrow \sqrt{r^2 - x^2} = \frac{x^2}{\sqrt{r^2 - x^2}}$ $r^2 - x^2 = x^2$ $2x^2 = r^2$ $x^2 = \frac{r^2}{2}$ $x = \frac{r}{\sqrt{2}}$		Correct derivative.	Correct solution presented in a correct mathematical manner.

NØ	N1	N2	A3	A4	M5	M6	E7	E8
No response; no relevant evidence.	ONE answer demonstrating limited knowledge of differentiation techniques.	1u	2u	3u	1r	2r	1t with minor error(s).	1t

(e) For the function $y = e^x \cos kx$:

(i) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

(ii) Find all the value(s) of k such that the function $y = e^x \cos kx$ satisfies the equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0 \text{ for all values of } x.$$

ASSESSOR'S USE ONLY

(e)	(i) $\frac{dy}{dx} = e^x.\cos kx + e^x(-k\sin kx)$ $= e^x(\cos kx - k\sin kx)$ $\frac{d^2y}{dx^2} = e^x(\cos kx - k\sin kx)$ $+e^x(-k\sin kx - k^2\cos kx)$ $= e^x(\cos kx - 2k\sin kx - k^2\cos kx)$ (ii) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0.$ $\Rightarrow e^x(\cos kx - 2k\sin kx - k^2\cos kx)$ $-2e^x(\cos kx - k\sin kx) + 2e^x\cos kx = 0$ $\Rightarrow e^x(\cos kx - k^2\cos kx) = 0$ $e^x\cos kx(1-k^2) = 0$ $k = \pm 1$	Correct expression for $\frac{dy}{dx}$	Correct expression for $\frac{d^2y}{dx^2}$	Correct solution with correct derivatives.

NØ	N1	N2	A3	A4	M5	M6	E7	E8
No response; no relevant evidence.	ONE answer demonstrating limited knowledge of differentiation techniques.	1u	2u	3u	1r	2r	1t with minor error(s).	1t

Cut Scores

Not Achieved	Achievement	Achievement with Merit	Achievement with Excellence
0 – 7	8 – 13	14 – 19	20 – 24

(e) A curve is defined by the parametric equations

$$x = t^3 + 1$$
$$y = t^2 + 1$$

Show that $\dfrac{\dfrac{d^2y}{dx^2}}{\left(\dfrac{dy}{dx}\right)^4}$ is a constant.

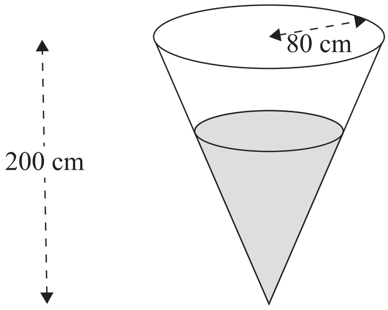
ASSESSOR'S USE ONLY

(d)	$\frac{dL}{dt} = 0.6 \text{ m s}^{-1}$ $L^2 = x^2 + 3^2$ $x = \sqrt{L^2 - 9}$ $\frac{dx}{dL} = \frac{1}{2}(L^2 - 9)^{\frac{-1}{2}} \cdot 2L$ $= \frac{L}{\sqrt{L^2 - 9}}$ $\frac{dx}{dt} = \frac{dL}{dt} \times \frac{dx}{dL}$ $= 0.6 \times \frac{L}{\sqrt{L^2 - 9}}$ When $L = 5.4$ $\frac{dx}{dt} = 0.6 \times \frac{5.4}{\sqrt{5.4^2 - 9}}$ $= 0.722 \text{ m s}^{-1}$	Correct expression for $\frac{dx}{dL}$ or $\frac{dL}{dx}$.	Correct solution with correct derivatives.	
(e)	$\frac{dx}{dt} = 3t^2 \quad \frac{dy}{dt} = 2t$ $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 2t \cdot \frac{1}{3t^2} = \frac{2}{3t}$ $\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dt} \times \frac{dt}{dx}$ $= \frac{-2}{3t^2} \times \frac{1}{3t^2} = \frac{-2}{9t^4}$ $\frac{\frac{d^2y}{dx^2}}{\left(\frac{dy}{dx}\right)^4} = \frac{\frac{-2}{9t^4}}{\left(\frac{2}{3t}\right)^4}$ $= \frac{-2}{9t^4} \times \frac{81t^4}{16}$ $= \frac{-9}{8}$ or -1.125	Correct $\frac{dy}{dx}$	Correct $\frac{d^2y}{dx^2}$	Correct solution with correct derivatives.

NØ	N1	N2	A3	A4	M5	M6	E7	E8
No response; no relevant evidence.	ONE answer demonstrating limited knowledge of differentiation techniques.	1u	2u	3u	1r	2r	1t with minor error(s).	1t

(e) A water tank is in the shape of an inverted right-circular cone.

The height of the cone is 200 cm and the radius of the cone is 80 cm.



The tank is being filled with water at a rate of 150 cm^3 per second.

At what rate will the surface area of the water in the tank be increasing when the depth of water in the tank is 125 cm?

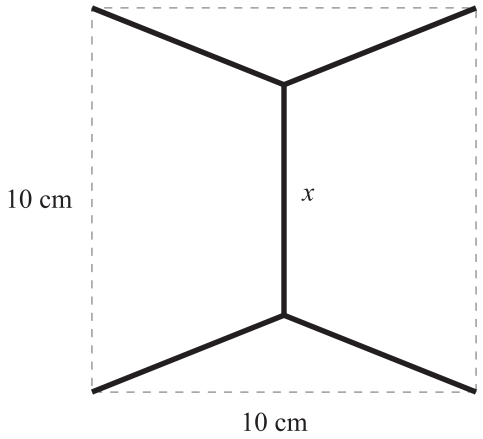
You must use calculus and show any derivatives that you need to find when solving this problem.

ASSESSOR'S USE ONLY

(e)	$\frac{dV}{dt} = 150\text{ cm}^3/\text{s}$ $\frac{dSA}{dt} = \frac{dV}{dt} \times \frac{dr}{dV} \times \frac{dSA}{dr}$ $h = 2.5r$ $V = \frac{1}{3}\pi r^2 h$ $= \frac{5}{6}\pi r^3$ $\frac{dV}{dr} = 2.5\pi r^2$ $SA = \pi r^2$ $\frac{dSA}{dr} = 2\pi r$ $\frac{dSA}{dt} = 150 \times \frac{1}{2.5\pi r^2} \times 2\pi r$ $= \frac{120}{r}$ When $h = 125\text{ cm}, r = 50\text{ cm}$ $\frac{dSA}{dt} = \frac{120}{50} = 2.4\text{ cm}^2/\text{s}$	Correct expression for $\frac{dV}{dr}$ in terms of one variable.	Correct expression for $\frac{dV}{dr}$ and $\frac{dSA}{dr}$ in terms of r, and an attempt to relate two (or more) derivatives.	Correct solution.

NØ	N1	N2	A3	A4	M5	M6	E7	E8
No response; no relevant evidence.	ONE answer demonstrating limited knowledge of differentiation techniques.	1u	2u	3u	1r	2r	1t with minor error(s).	1t

(e)



The above shape is made from wire. It has both vertical and horizontal lines of symmetry.

The ends of the shape are at the vertices of a square with a side length of 10 cm, as shown in the diagram above.

The length of the piece of wire through the centre of the shape is x cm.

Find the value(s) of x that enables the shape to be made with the minimum length of wire.

You do not need to prove that the length is a minimum.

You must use calculus and show any derivatives that you need to find when solving this problem.

ASSESSOR'S USE ONLY

(e)	$w^2 = 5^2 + \left(5 - \frac{x}{2}\right)^2$ $w^2 = 25 + 25 - 5x + 0.25x^2$ $w^2 = 0.25x^2 - 5x + 50$ $w = \left(0.25x^2 - 5x + 50\right)^{\frac{1}{2}}$ $\text{Length} = x + 4w$ $= x + 4\left(0.25x^2 - 5x + 50\right)^{\frac{1}{2}}$ $\frac{dL}{dx} = 1 + 2\left(0.25x^2 - 5x + 50\right)^{\frac{-1}{2}} \times (0.5x - 5)$ $\frac{dL}{dx} = 1 + \frac{x - 10}{\left(0.25x^2 - 5x + 50\right)^{\frac{1}{2}}}$ For max/min $\frac{dL}{dx} = 0$ $\frac{x - 10}{\left(0.25x^2 - 5x + 50\right)^{\frac{1}{2}}} = -1$ $x - 10 = -1\left(0.25x^2 - 5x + 50\right)^{\frac{1}{2}}$ $(x - 10)^2 = 0.25x^2 - 5x + 50$ $x^2 - 20x + 100 = 0.25x^2 - 5x + 50$ $0.75x^2 - 15x + 50 = 0$ $x = 15.77$ not applicable $x = 4.23\text{ cm}$		Correct expression for $\frac{dL}{dx}$	Correct solution with correct derivative.

NØ	N1	N2	A3	A4	M5	M6	E7	E8
No response; no relevant evidence.	ONE answer demonstrating limited knowledge of differentiation techniques.	1u	2u	3u	1r	2r	1t with minor error(s).	1t

Cut Scores

Not Achieved	Achievement	Achievement with Merit	Achievement with Excellence
0 – 7	8 – 12	13 – 18	19 – 24

ASSESSOR'S USE ONLY

(d) For what value(s) of x is the function $y = x^3e^x$ decreasing?

You must use calculus and show any derivatives that you need to find when solving this problem.

(e) The volume of a sphere is increasing.

At the instant when the sphere's radius is 0.5 m, the surface area of the sphere is increasing at a rate of 0.4 $m^2\ s^{-1}$.

Find the rate at which the volume of the sphere is increasing at this instant.

You must use calculus and show any derivatives that you need to find when solving this problem.

Assessment Schedule – 2019

Calculus: Apply differentiation methods in solving problems (91578)

Evidence Statement

Q1	Expected coverage	Achievement (u)	Merit (r)	Excellence (t)
(a)	$\frac{dy}{dx} = \frac{1}{2}\left(3x^2 - 1\right)^{\frac{-1}{2}}.6x$ $= \frac{3x}{\sqrt{3x^2 - 1}}$	Correct derivative. Anything equivalent.		
(b)	$f'(t) = \frac{15}{3t - 1}$ $f'(4) = \frac{15}{11}$ or 1.36	Correct solution with correct derivative.		
(c)	Quotient rule $\frac{dy}{dx} = \frac{\left(1 + x^2\right)2e^{2x} - e^{2x}\left(2x\right)}{\left(1 + x^2\right)^2}$ OR Product rule $\frac{dy}{dx} = e^{2x}\left(-2x\right)\left(1 + x^2\right)^{-2} + \left(1 + x^2\right)^{-1}\left(2e^{2x}\right)$ When $x = 2$, $\frac{dy}{dx} = \frac{6e^4}{25}$ or 13.1	Correct derivative.	Correct solution with correct derivative.	
(d)	$\frac{dy}{dx} = 3x^2e^x + x^3e^x$ $= x^2e^x\left(3 + x\right)$ $\frac{dy}{dx} < 0$ $\Rightarrow x^2e^x\left(3 + x\right) < 0$ $3 + x < 0$ $x < -3$	Correct derivative.	Correct solution with correct derivative.	
(e)	$\frac{dV}{dt} = \frac{dS}{dt} \times \frac{dr}{dS} \times \frac{dV}{dr}$ $S = 4\pi r^2 \Rightarrow \frac{dS}{dr} = 8\pi r$ $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dr} = 4\pi r^2$ $\frac{dS}{dt} = 0.4$ when $r = 0.5$ $\frac{dV}{dt} = 0.4 \times \frac{1}{8\pi r} \times 4\pi r^2$ $= 0.2r$ When $r = 0.5$, $\frac{dV}{dt} = 0.1\,m^3 / s$	Correct expressions for $\frac{dS}{dr}$ and $\frac{dV}{dr}$.	Correct expression for $\frac{dV}{dt}$. Anything equivalent. Line 5 is ok.	Correct solution with correct derivatives. Units not required.

(e) If $y = e^u$ and $u = \sin 2x$ show that

$$\frac{d^2y}{dx^2} = \frac{d^2y}{du^2}\left(\frac{du}{dx}\right)^2 + \frac{dy}{du}\frac{d^2u}{dx^2}$$

You must use calculus and show any derivatives that you need to find when solving this problem.

ASSESSOR'S USE ONLY

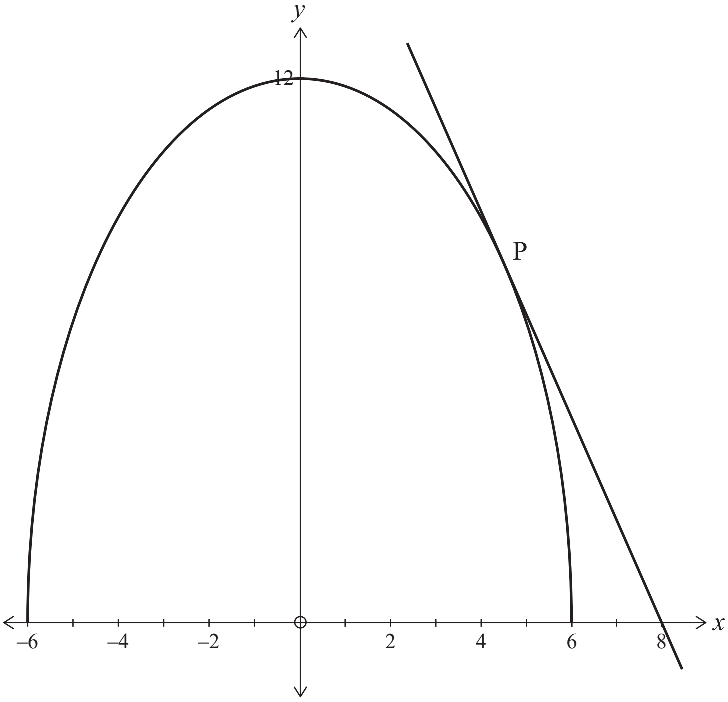
(e)	LHS $y = e^{\sin 2x}$ $\frac{dy}{dx} = e^{\sin 2x} \times 2\cos 2x$ $\frac{d^2y}{dx^2} = e^{\sin 2x} \times (-4\sin 2x) + e^{\sin 2x} \times (2\cos 2x)^2$ $u = \sin 2x$ $\frac{du}{dx} = 2\cos 2x$ $\frac{d^2u}{dx^2} = -4\sin 2x$ $y = e^u$ $\frac{dy}{du} = e^u$ $\frac{d^2y}{du^2} = e^u$ RHS $\frac{d^2y}{du^2} \times \left(\frac{du}{dx}\right)^2 + \frac{dy}{du} \times \frac{d^2u}{dx^2}$ $= e^u \times (2\cos 2x)^2 + e^u \times (-4\sin 2x)$ $= e^{\sin x} \times (2\cos 2x)^2 + e^{\sin x} \times (-4\sin 2x)$ Therefore LHS = RHS as required. $\frac{d^2y}{dx^2} = 4e^{\sin 2x}(\cos^2 2x - \sin 2x)$	Correct expression for $\frac{dy}{dx}$ or $\frac{du}{dx}$.	Correct expressions for $\frac{d^2y}{dx^2}$ in any equivalent form. Or correct RHS.	Complete proof. Accept in terms of x, y, and u.

NØ	N1	N2	A3	A4	M5	M6	E7	E8
No response; no relevant evidence.	ONE answer demonstrating limited knowledge of differentiation techniques.	1u	2u	3u	1r	2r	1t with minor error(s).	1t

ASSESSOR'S USE ONLY

(e) The graph below shows the function $y = 2\sqrt{36 - x^2}$, and the tangent to that function at point P.

The tangent intersects the x-axis at the point (8,0).

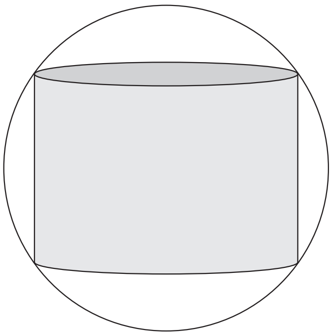


Find the x-coordinate of point P.

You must use calculus and show any derivatives that you need to find when solving this problem.

(e)	$y = 2\sqrt{36 - x^2}$ $\frac{dy}{dx} = \left(36 - x^2\right)^{\frac{-1}{2}} . -2x$ $= \frac{-2x}{\sqrt{36 - x^2}}$ Gradient of tangent $= \frac{-2\sqrt{36 - x^2}}{(8 - x)}$ $= \frac{2\sqrt{36 - x^2}}{x - 8}$ $\therefore \frac{2\sqrt{36 - x^2}}{x - 8} = \frac{-2x}{\sqrt{36 - x^2}}$ $2\left(36 - x^2\right) = 16x - 2x^2$ $72 - 2x^2 = 16x - 2x^2$ $72 = 16x$ $x = 4.5$ Or alternatively: $y = \frac{-2x}{\sqrt{36 - x^2}}(x - 8)$ $y = \frac{-2x}{\sqrt{36 - x^2}} + \frac{16x}{\sqrt{36 - x^2}}$ Substituting for y: $2\sqrt{36 - x^2} = \frac{-2x^2}{\sqrt{36 - x^2}} + \frac{16x}{\sqrt{36 - x^2}}$ $2\left(36 - x^2\right) = -2x^2 + 16x$ $36 - x^2 = -x^2 + 8x$ $36 = 8x$ $x = 4.5$	Correct $\frac{dy}{dx}$ of curve.	Correct $\frac{dy}{dx}$ of curve. AND Correct gradient of tangent. OR Correct equation of tangent involving expression for $\frac{dy}{dx}$.	Correct solution with correct derivatives.

(e) A cylinder of height h and radius r is inscribed, as shown to the right, inside a sphere of radius 20 cm.

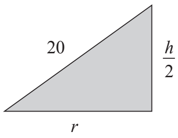


Find the maximum possible volume of the cylinder.

You must use calculus and show any derivatives that you need to find when solving this problem.

You do not need to prove that the volume you have found is a maximum.

ASSESSOR'S USE ONLY

	Expected Coverage	Achievement (u)	Merit (r)	Excellence (t)
(e)	$r^2+\left(\frac{h}{2}\right)^2=400$ $r^2=400-\frac{h^2}{4}$ $V_{\text{cyl}}=\pi r^2 h$ $=\pi\left(400-\frac{h^2}{4}\right)h$ $=\pi\left(400h-\frac{h^3}{4}\right)$ $\frac{\mathrm{d}V}{\mathrm{d}h}=\pi\left(400-\frac{3h^2}{4}\right)$ $\frac{\mathrm{d}V}{\mathrm{d}h}=0\Rightarrow 400-\frac{3h^2}{4}=0$ $h=\sqrt{\frac{1600}{3}}=\frac{40}{\sqrt{3}}=23.1\text{ cm}$ $r=16.3\text{ cm}$ $V=\pi\times16.3^2\times23.1$ $=19\,300\text{ cm}^3$ $V=19\,347\text{ cm}^3$  	Correct expression for $\frac{\mathrm{d}V}{\mathrm{d}h}$ or $\frac{\mathrm{d}V}{\mathrm{d}r}$	Correct value of r or h with correct derivatives. Units not required.	Correct solution with correct derivatives. Units not required.

NØ	N1	N2	A3	A4	M5	M6	E7	E8
No response; no relevant evidence.	ONE answer demonstrating limited knowledge of differentiation techniques.	1u	2u	3u	1r	2r	1t with one minor error	1t

ASSESSOR'S USE ONLY

(e) A curve is defined by the parametric equations $x = \ln(t)$ and $y = 6t^3$ where $t > 0$.

The point P lies on the curve, and at point P, $\frac{d^2y}{dx^2} = 2$.

Find the exact coordinates of point P.

You must use calculus and show any derivatives that you need to find when solving this problem.

	Expected Coverage	Achievement (u)	Merit (r)	Excellence (t)
(e)	$\frac{dx}{dt} = \frac{1}{t} \quad \frac{dy}{dt} = 18t^2$ $\frac{dy}{dx} = 18t^3$ $\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right).\frac{dt}{dx}$ $= 54t^2 \times t$ $= 54t^3$ $54t^3 = 2$ $t^3 = \frac{1}{27}$ $t = \frac{1}{3}$ $x = \ln\left(\frac{1}{3}\right)$ $y = 6\left(\frac{1}{3}\right)^3$ $= \frac{2}{9}$ P is $\left(\ln\left(\frac{1}{3}\right), \frac{2}{9}\right)$	Correct expression for $\frac{dy}{dx}$.	Correct expression for $\frac{d^2y}{dx^2}$.	Correct solution with correct derivatives. Accept (–1.1, 0.22).

NØ	N1	N2	A3	A4	M5	M6	E7	E8
No response; no relevant evidence.	ONE answer demonstrating limited knowledge of differentiation techniques.	1u	2u	3u	1r	2r	1t	2t

ASSESSOR'S USE ONLY

(e) A curve has the equation $y = (3x+2)e^{-2x}$.

Prove that $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$.

You must use calculus and show any derivatives that you need to find when solving this problem.

	Expected Coverage	Achievement (u)	Merit (r)	Excellence (t)
(e)	$\frac{dy}{dx} = (3x+2)e^{-2x}.(-2) + 3e^{-2x}$ $= e^{-2x}[-2(3x+2)+3]$ $= e^{-2x}(-6x-1)$ $\frac{d^2y}{dx^2} = -6e^{-2x} - 2e^{-2x}(-6x-1)$ $= e^{-2x}[-6-2(-6x-1)]$ $= e^{-2x}(-6+12x+2)$ $= e^{-2x}(12x-4)$ $= 4e^{-2x}(3x-1)$ EITHER $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$ LHS$= 4e^{-2x}(3x-1) + 4e^{-2x}(-6x-1) + 4e^{-2x}(3x+2)$ $= 4e^{-2x}[3x-1-6x-1+3x+2]$ $= 0$ $=$ RHS as required OR LHS$= e^{-2x}(12x-4) + 4e^{-2x}(-6x-1) + 4e^{-2x}(3x+2)$ $= e^{-2x}[12x-4+4(-6x-1)+4(3x+2)]$ $= e^{-2x}[12x-4+24x-4+12x+8]$ $= 0$ $=$ RHS as required	Correct expression for $\frac{dy}{dx}$.	Correct expression for $\frac{d^2y}{dx^2}$.	Correct solution with correct derivatives.

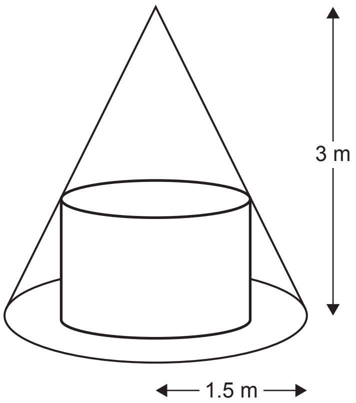
NØ	N1	N2	A3	A4	M5	M6	E7	E8
No response; no relevant evidence.	ONE answer demonstrating limited knowledge of differentiation techniques.	1u	2u	3u	1r	2r	1t	2t

Cut Scores

Not Achieved	Achievement	Achievement with Merit	Achievement with Excellence
0 – 8	9 – 14	15 – 20	21 – 24

(e) A cone has a height of 3 m and a radius of 1.5 m.

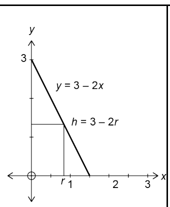
A cylinder is inscribed in the cone, as shown in the diagram below.



The base of the cylinder has the same centre as the base of the cone.

Prove that the maximum volume of the cylinder is π m^3.

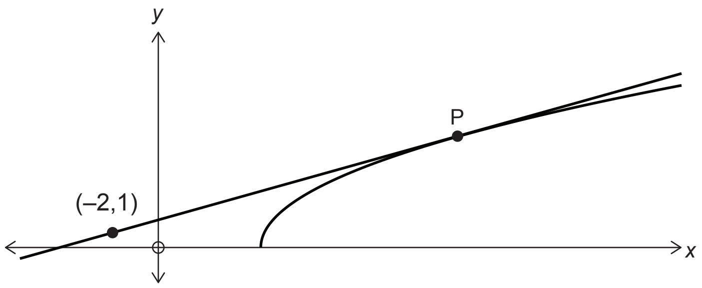
You must use calculus and show any derivatives that you need to find when solving this problem.

(e)	$V = \pi r^2 h$ $= \pi r^2 (3 - 2r)$ $= 3\pi r^2 - 2\pi r^3$ $\frac{dV}{dr} = 6\pi r - 6\pi r^2$ At maximum, $\frac{dV}{dr} = 0$ $6\pi r(1 - r) = 0$ $r = 0$ (no) $\therefore r = 1$ $V = \pi 1^2 (3 - 2 \times 1) = \pi$ $\frac{d^2V}{dr^2} = 6\pi - 12\pi r$ When $r = 1$, $\frac{d^2V}{dr^2} = -6\pi < 0$ Therefore $V = \pi$ is maximum volume.  	Correct expression for $\frac{dV}{dr}$.	Correct expression for $\frac{dV}{dr}$ and finds $r = 1$.	**T1**: Correct expression for $\frac{dV}{dr}$ and shows that $V = \pi$ but does not prove it is the maximum volume with either the first or second derivative test. **T2**: Correct expression for $\frac{dV}{dr}$ and correct proof.

NØ	N1	N2	A3	A4	M5	M6	E7	E8
No response; no relevant evidence.	ONE partial solution.	1u	2u	3u	1r	2r	T1	T2 or two T1

(e) The graph below shows the curve $y = \sqrt{2x - 4}$, and the tangent to the curve at point P.

The tangent passes through the point (–2,1).



Find the coordinates of point P.

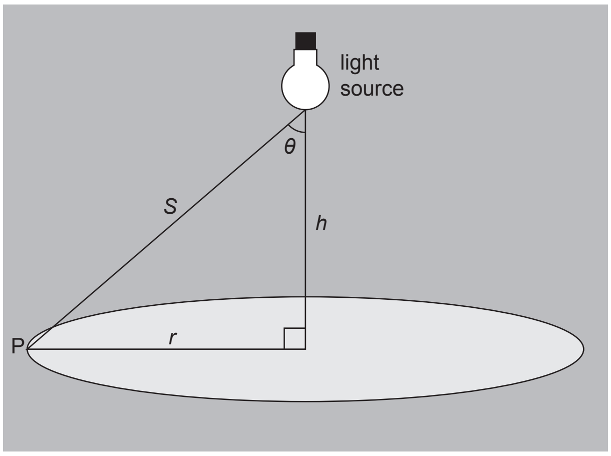
You must use calculus and show any derivatives that you need to find when solving this problem.

(e)	$y=\sqrt{2x-4}$ $\frac{dy}{dx}=\frac{1}{\sqrt{2x-4}}$ Gradient of tangent $=\frac{y-1}{x+2}$ $\frac{1}{\sqrt{2x-4}}=\frac{\sqrt{2x-4}-1}{x+2}$ $x+2=2x-4-\sqrt{2x-4}$ $\sqrt{2x-4}=x-6$ $2x-4=x^2-12x+36$ $x^2-14x+40=0$ $(x-4)(x-10)=0$ $x=4$ or $x=10$ Rejecting $x=4$ by checking the surd equation $x=10 \quad \sqrt{16}=4 \quad$ True $x=4 \quad \sqrt{4}=-2 \quad$ False One solution: $x=10$ Therefore, the coordinates of point P are $(10,4)$ OR Rejecting $x=4$ by checking the gradient: At $(10,4)$, $\frac{dy}{dx}=\frac{1}{\sqrt{16}}=\frac{1}{4}$ Gradient: $\frac{y-1}{x+2}=\frac{3}{12}=\frac{1}{4}$ At $(4,2)$, $\frac{dy}{dx}=\frac{1}{\sqrt{4}}=\frac{1}{2}$ Gradient: $\frac{y-1}{x+2}=\frac{1}{6}$ One solution: $x=10$ Therefore, the coordinates of point P are $(10,4)$	Correct derivative: $\frac{dy}{dx}=\frac{1}{\sqrt{2x-4}}$	Correct derivative: $\frac{dy}{dx}=\frac{1}{\sqrt{2x-4}}$ and $\sqrt{2x-4}=x-6$	**T1:** Correct solution with correct derivative: P (10,4) without any justification for $x \neq 4$ **T2:** Correct solution with correct derivative: P (10,4) $x \neq 4$ must be justified with respect to either the surd equation or the gradient of the tangent.

NØ	N1	N2	A3	A4	M5	M6	E7	E8
No response; no relevant evidence.	ONE partial solution.	1u	2u	3u	1r	2r	T1	T2

(e) A lamp is suspended above the centre of a round table of radius r.

The height, h, of the lamp above the table is adjustable.



Point P is on the edge of the table.

At point P the illumination I is directly proportional to the cosine of angle θ in the above diagram, and inversely proportional to the square of the distance, S, to the lamp.

i.e. $I = \dfrac{k\cos\theta}{S^2}$, where k is a constant.

Prove that the edge of the table will have maximum illumination when $h = \dfrac{r}{\sqrt{2}}$.

You do not need to prove that your solution gives the maximum value.

You must use calculus and show any derivatives that you need to find when solving this problem.

Year 2021 - Question 3(e) - ANSWER

(e)	$\cos\theta = \frac{h}{S}$ $S^2 = h^2 + r^2$ $S = \sqrt{h^2 + r^2}$ k and r are constant $I = \frac{k\cos\theta}{S^2}$ $I = \frac{k\frac{h}{S}}{S^2}$ $= \frac{kh}{S^3}$ $I = \frac{kh}{\left(h^2 + r^2\right)^{\frac{3}{2}}}$ $\frac{\mathrm{d}I}{\mathrm{d}h} = \frac{\left(h^2 + r^2\right)^{\frac{3}{2}}k - kh\left(\frac{3}{2}\right)\left(h^2 + r^2\right)^{\frac{1}{2}}(2h)}{\left(h^2 + r^2\right)^3}$ $\frac{\mathrm{d}I}{\mathrm{d}h} = \frac{k\left(h^2 + r^2\right)^{\frac{3}{2}} - 3kh^2\left(h^2 + r^2\right)^{\frac{1}{2}}}{\left(h^2 + r^2\right)^3}$ $\frac{\mathrm{d}I}{\mathrm{d}h} = \frac{k\left(h^2 + r^2\right)^{\frac{1}{2}}\left(h^2 + r^2 - 3h^2\right)}{\left(h^2 + r^2\right)^3}$ $\frac{\mathrm{d}I}{\mathrm{d}h} = \frac{k\left(r^2 - 2h^2\right)}{\left(h^2 + r^2\right)^{\frac{5}{2}}}$ $\frac{\mathrm{d}I}{\mathrm{d}h} = 0 \Rightarrow k\left(r^2 - 2h^2\right) = 0$ $2h^2 = r^2$ $h^2 = \frac{r^2}{2}$ $h = \frac{r}{\sqrt{2}}$		Correct expression for $\frac{\mathrm{d}I}{\mathrm{d}h}$	T2: Correct proof with correct derivative

NØ	N1	N2	A3	A4	M5	M6	E7	E8
No response; no relevant evidence.	ONE partial solution.	1u	2u	3u	1r	2r	T1	T2

Cut Scores

Not Achieved	Achievement	Achievement with Merit	Achievement with Excellence
0 – 6	7– 12	13 – 18	19 – 24

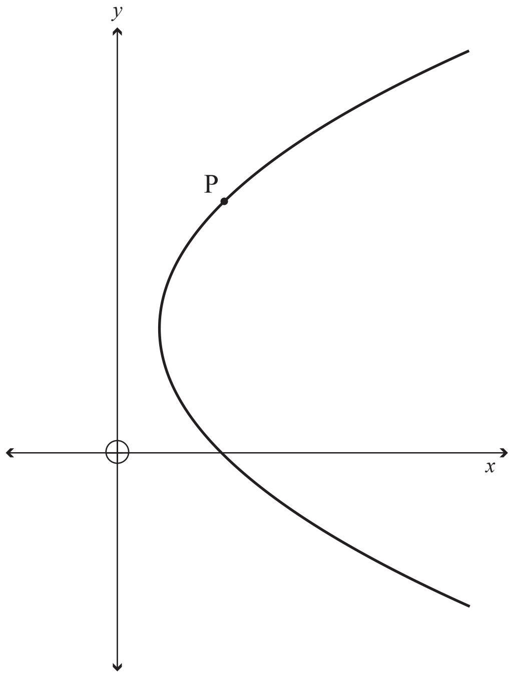
(e) If p is a positive real constant, prove that $y = e^{px^2}$ does not have any points of inflection.

You must use calculus and show any derivatives that you need to find when solving this problem.

Year 2022 - Question 1(e) - ANSWER

	Expected coverage	Achievement (u)	Merit (r)	Excellence (t)
(e)	$y = e^{px^2}$ $\frac{dy}{dx} = 2pxe^{px^2}$ $\frac{d^2y}{dx^2} = 2px.2px.e^{px^2} + 2p.e^{px^2}$ $= 2pe^{px^2}(2px^2 + 1)$ At a point of inflection, $\frac{d^2y}{dx^2} = 0$ $2pe^{px^2}(2px^2 + 1) = 0$ **Equation 1** $2pe^{px^2} = 0$ $pe^{px^2} = 0$ $px^2 = \ln 0$ No solution as ln 0 is defined. OR $2pe^{px^2} = 0$ has no solutions because $2pe^{px^2} > 0$ for all values of x since $e^{px^2} > 0$ for all values of a and p is a positive **Equation 2** $2px^2 + 1 = 0$ $x^2 = \frac{-1}{2p}$ $x = \sqrt{\frac{-1}{2p}}$ $2px^2 + 1 = 0$ has no real solutions because p is a positive real constant, $\frac{-1}{2p}$ is negative and there is not a real solution when you take the square root of a negative number. OR $2px^2 + 1 = 0$ has no real solutions because $2px^2 + 1 = 0$ is always greater than zero because p is a positive real constant and x^2 is always greater than or equal to zero OR $2px^2 + 1 = 0$ has no real solutions because the discriminant is less than zero. $b^2 - 4ac = 0 - 4(2p)(1) = -8p$ Since p is a positive real constant. Therefore, there are no solutions to $\frac{d^2y}{dx^2} = 0$ nd and $y = e^{px^2}$ has no points of inflection.	Correct $\frac{dy}{dx}$.	Correct $\frac{d^2y}{dx^2}$.	**T1** Correct $\frac{d^2y}{dx^2}$ with one part of the equation set to zero and the reason for there being no real solutions given for EITHER $2pe^{px^2} = 0$ OR $2px^2 + 1 = 0$ **T2** Correct proof with correct derivatives Both parts of the equation set to zero and the reason for there being no real solutions given for both equations

(e) The curve with the equation $(y-5)^2 = 16(x-2)$ has a tangent of gradient 1 at point P.



This tangent intersects the x and y axes at points R and S respectively.

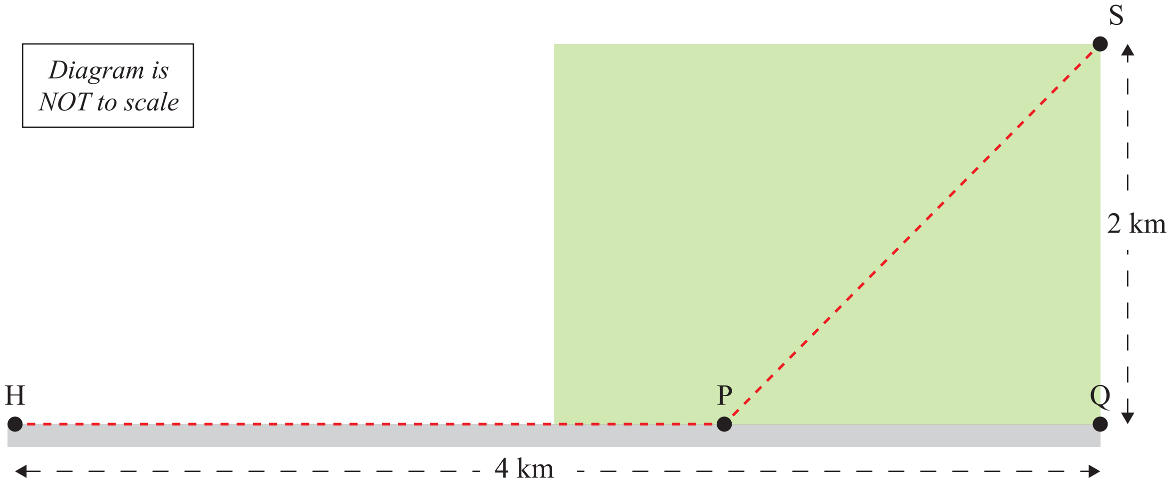
Prove that the length RS is $7\sqrt{2}$.

You must use calculus and show any derivatives that you need to find when solving this problem.

	Expected coverage	Achievement (u)	Merit (r)	Excellence (t)
(e)	$(y-5)^2 = 16(x-2)$ Method A $y-5 = 4\sqrt{x-2}$ $y = 4\sqrt{x-2}+5$ $\frac{dy}{dx} = \frac{2}{\sqrt{x-2}}$ $\frac{dy}{dx} = 1$ $\frac{2}{\sqrt{x-2}} = 1$ $\sqrt{x-2} = 2$ $x-2 = 4$ $x = 6$ $y = 13$ Method B $2(y-5)\frac{dy}{dx} = 16$ $\frac{dy}{dx} = \frac{8}{y-5}$ $\frac{dy}{dx} = 1$ $\frac{8}{y-5} = 1$ $y = 13$ $64 = 16x - 32$ $x = 6$ Equation of tangent $y - 13 = 1(x-6)$ $y = x + 7$ Axis intercepts: (0,7) and (-7, 0) Distance RS = $\sqrt{7^2 + 7^2}$ $= \sqrt{49 \times 2}$ $= 7\sqrt{2}$	Correct $\frac{dy}{dx}$.	Correct x and y values found (6,13) with correct $\frac{dy}{dx}$.	**T1** Finds equation of tangent and both axis intercepts with correct $\frac{dy}{dx}$. **T2** Correct solution with correct $\frac{dy}{dx}$.

NØ	N1	N2	A3	A4	M5	M6	E7	E8
No response; no relevant evidence.	ONE partial solution.	1u	2u	3u	1r	2r	T1	T2

(e) Megan cycles from her home, H, to school, S, each day.



She rides along a path from her home to point P at a constant speed of 10 kilometres per hour.

At point P, Megan cuts across a park, heading directly to school. When cycling across the park, Megan can only cycle at 6 kilometres per hour.

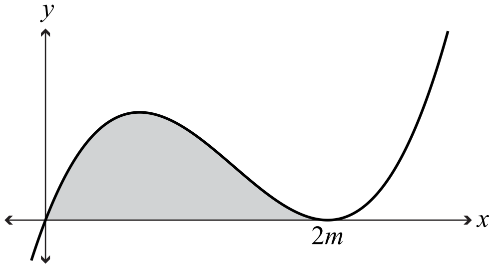
At what distance from her home should she choose to cut across the park in order to make her travelling time a minimum?

You must use calculus and show any derivatives that you need to find when solving this problem.

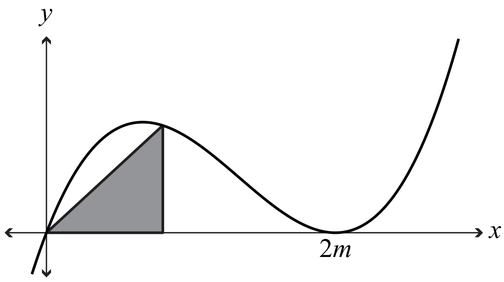
	Expected coverage	Achievement (u)	Merit (r)	Excellence (t)
(e)	Total time = time (HP) + time (PS) **Method A** Let x = distance PQ $T=\frac{4-x}{10}+\frac{\sqrt{x^2+4}}{6}$ $\frac{dT}{dx}=\frac{-1}{10}+\frac{\frac{1}{2}(x^2+4)^{\frac{-1}{2}}.2x}{6}$ $\frac{dT}{dx}=\frac{-1}{10}+\frac{x}{6\sqrt{x^2+4}}$ For maximum/minimum time, $\frac{dT}{dx}=0$ $\frac{1}{10}=\frac{x}{6\sqrt{x^2+4}}$ $6\sqrt{x^2+4}=10x$ $\sqrt{x^2+4}=\frac{10}{6}x$ $x^2+4=\frac{25}{9}x^2$ $4=\frac{16}{9}x^2$ $\frac{36}{16}=x^2$ $x=1.5$ $4-1.5=2.5$ Megan should travel 2.5 km along the path before cutting across the park. **Method B** Let x = distance HP $T=\frac{x}{10}+\frac{\sqrt{\left((4-x)^2+4\right)}}{6}$ $\frac{dT}{dx}=\frac{1}{10}+\frac{(x-4)}{6\sqrt{x^2-8x+20}}$ $\frac{dT}{dx}=0$ $\frac{1}{10}+\frac{(x-4)}{6\sqrt{x^2-8x+20}}=0$ $5(x-4)=-3\sqrt{x^2-8x+20}$ $25\left(x^2-8x+16\right)=9\left(x^2-8x+20\right)$ $25x^2-200x+400=9x^2-72x+180$ $16x^2-128x+220=0$ $x=2.5$ or 5.5 Since $x<4$, $x=2.5$ km		Correct $\frac{dT}{dx}$.	**T1** Method A $x = 1.5$ found with correct derivative OR **T1** Method B $x = 2.5$ or 5.5 found (5.5 not discarded) with correct derivative. **T2** Correct solution with correct derivative.

NØ	N1	N2	A3	A4	M5	M6	E7	E8
No response; no relevant evidence.	ONE partial solution.	1u	2u	3u	1r	2r	T1	T2 or two of T1

(e) The graph of $y = x(x - 2m)^2$, where $m > 0$, is shown. The total shaded area between the curve and the x-axis from $x = 0$ to $x = 2m$ is given by $A = \dfrac{4m^4}{3}$.



A right-angled triangle is now constructed with one vertex at (0,0) and another on the curve $y = x(x - 2m)^2$, as shown below.



Show that the maximum area of such a triangle is $\dfrac{3}{8}$ of the total shaded area.

You must use calculus and show any derivatives that you need to find when solving this problem.

You do not have to prove that the area you have found is a maximum.

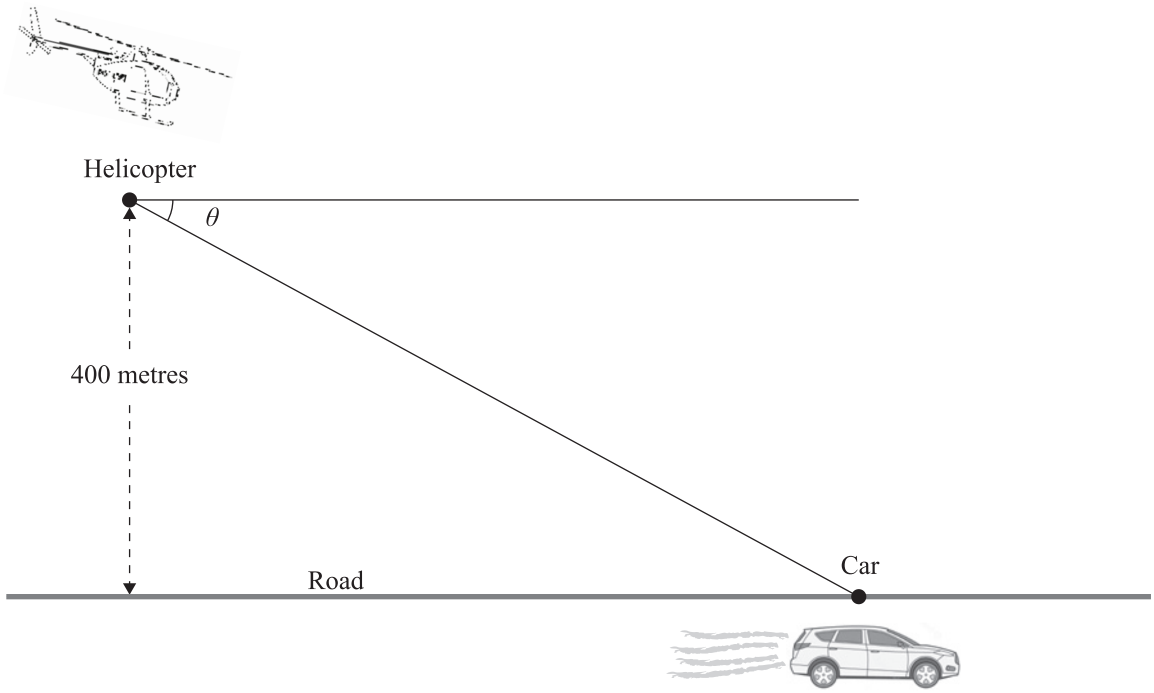
(e)	Area of triangle $= \frac{1}{2}xy$ $A = \frac{1}{2}x\left(x(x-2m)^2\right)$ $= \frac{1}{2}x^2(x-2m)^2$ $\frac{\mathrm{d}A}{\mathrm{d}x} = \frac{1}{2}x^2(2(x-2m)) + (x-2m)^2$ $= x^2(x-2m) + x(x-2m)^2$ $= x(x-2m)\left(x+(x-2m)\right)$ $= x(x-2m)(2x-2m)$ $= 2x(x-2m)(x-m)$ OR $A = \frac{1}{2}x^2(x-2m)^2$ $= \frac{1}{2}x^4 - 2mx^3 + 2m^2x^2$ $\frac{\mathrm{d}A}{\mathrm{d}x} = 2x^3 - 6mx^2 + 4m^2x$ $= 2x\left(x^2 - 3mx + 2m^2\right)$ $= 2x(x-2m)(x-m)$ $\frac{\mathrm{d}A}{\mathrm{d}x} = 0 \Rightarrow 2x(x-2m)(x-m) = 0$ $x = 0$ or $x = 2m$ or $x = m$ Since $0 < x < 2m$ the area is a maximum when $x = m$ Maximum area of triangle: $A(m) = \frac{1}{2}m^2\left(m-2m\right)^2$ $= \frac{1}{2}m^4$ This is $\frac{3}{8}$ of the total shaded area since $\frac{3}{8} \times \frac{4m^3}{3} = \frac{1}{2}m^4$	• Correct derivative.	• Correct derivative. AND $x = m$ found.	T1 Maximum area, $A = \frac{1}{2}m^4$ found with correct $\frac{\mathrm{d}A}{\mathrm{d}x}$. OR Correct solution but with one minor error. T2 Correct solution with correct $\frac{\mathrm{d}A}{\mathrm{d}x}$ showing the calculation of the correct proportion of total shaded area..

NØ	N1	N2	A3	A4	M5	M6	E7	E8
No response; no relevant evidence.	ONE partial solution.	1u	2u	3u	1r	2r	1t	2t

(e) A police helicopter is flying above a straight horizontal section of motorway chasing a speeding car.

The helicopter is flying at a constant speed of 72 m s^{-1} and at a constant height of 400 metres above the ground. The helicopter is attempting to catch up with the car.

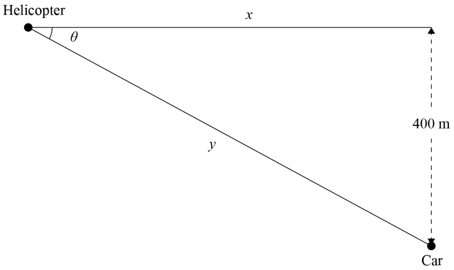
When the direct distance from the helicopter to the car is 2500 metres, the angle of depression, θ, between the horizontal and the line of sight from the helicopter to the car is increasing at a rate of 0.002 rad s^{-1}.



Adapted from: https://animalia-life.club/qa/pictures/police-helicopter-drawing, https://www.freepik.com/premium-vector/car-sedan-suv-drawing-outlines-converted-objects_17981778.htm

Calculate the speed of the **car** at this instant.

You must use calculus and show any derivatives that you need to find when solving this problem.

(e)	 Let x = horizontal distance between the helicopter and the car. Let y = direct distance between the helicopter and the car. Given: $\frac{d\theta}{dt} = 0.002\ rad\ s-1$ $\tan\theta = \frac{400}{x}$ $x = 400\cot\theta$ $\frac{dx}{d\theta} = -400\operatorname{cosec}^2\theta$ $= \frac{-400}{\sin^2\theta}$ $\frac{dx}{dt} = \frac{dx}{d\theta} \times \frac{d\theta}{dt}$ $= \frac{-400}{\sin^2\theta} \times 0.002$ $= \frac{-0.8}{\sin^2\theta}$ When $y = 2500$, $\sin\theta = \frac{400}{2500}$ $\theta = 0.1607$ rad $\frac{dx}{dt} = \frac{-0.8}{\sin^2(0.1607}$ $= 31.25$ When the helicopter is travelling at 72 m s^{-1}, The speed of the car $= 72 - 31.25$ $= 40.75$ m s^{-1} $(= 146.7$ km/hr$)$	• Finds $\frac{dx}{d\theta}$. .	• Finds an expression for $\frac{dx}{dt}$.	T1 Finds the value for $\frac{dx}{dt} = -31.25$ With correct derivatives. OR Finds correct solution but with one minor error. T2 Finds $\frac{dx}{dt} = -31.25$ with correct derivatives. AND The speed of the car = 40.76 m s^{-1}.

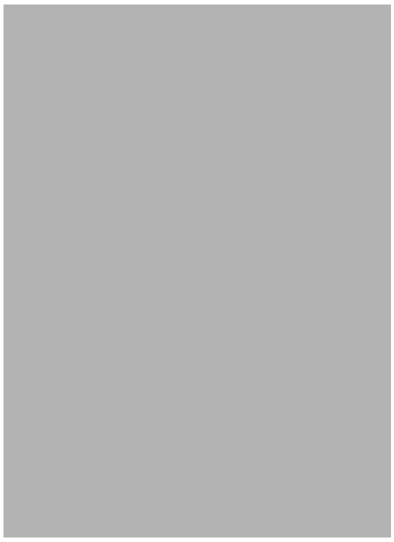
NØ	N1	N2	A3	A4	M5	M6	E7	E8
No response; no relevant evidence.	ONE partial solution.	1u	2u	3u	1r	2r	1t	2t

(e) A power line hangs between two poles.

The equation of the curve $y = f(x)$ that models the shape of the power line can be found by solving the differential equation:

$$a\frac{d^2y}{dx^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

Use differentiation to verify that the function $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$ satisfies the above differential equation, where a is a positive constant.



Source: www.thelocalelectrician.com.au/power-lines-power-pole-who-is-responsible/

(e)	$y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{\frac{-x}{a}}\right)$ $= \left(\frac{a}{2}e^{\frac{x}{a}} + \frac{a}{2}e^{\frac{-x}{a}}\right)$ $\frac{dy}{dx} = \frac{1}{2}e^{\frac{x}{a}} - \frac{1}{2}e^{\frac{-x}{a}}$ $\frac{d^2y}{dx^2} = \frac{1}{2a}e^{\frac{x}{a}} + \frac{1}{2a}e^{\frac{-x}{a}}$ $\left(\frac{dy}{dx}\right)^2 = \frac{1}{4}e^{\frac{2x}{a}} + \frac{1}{4}e^{\frac{-2x}{a}} - \frac{1}{2}$ #(1) $\text{LHS} = a\frac{d^2y}{dx^2}$ $= \frac{1}{2}e^{\frac{x}{a}} + \frac{1}{2}e^{\frac{-x}{a}}$ #(2) $\text{RHS} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$ $= \sqrt{1 + \frac{1}{4}e^{\frac{2x}{a}} + \frac{1}{4}e^{\frac{-2x}{a}} - \frac{1}{2}}$ $= \sqrt{\frac{1}{4}e^{\frac{2x}{a}} + \frac{1}{4}e^{\frac{-2x}{a}} + \frac{1}{2}}$ #(3) $= \sqrt{\left(\frac{1}{4}e^{\frac{2x}{a}} + e^{\frac{-2x}{a}}\right)^2}$ $= \frac{1}{2}\left(e^{\frac{x}{a}} - e^{\frac{-x}{a}}\right)$ $= a\frac{d^2y}{dx^2}$ $=$ LHS as required	• Correct expression for $\frac{dy}{dx}$.	• Correct expression for $\frac{d^2y}{dx^2}$. AND Evidence of progress with substitution into the differential equation Reaches either stage #(1) OR stage #(2).	T1 • Reaches both stage #(2) AND stage #(3) with correct derivatives. OR Correct solution but with one minor error. T2 • Correct proof with correct derivatives.

NØ	N1	N2	A3	A4	M5	M6	E7	E8
No response; no relevant evidence.	ONE partial solution.	1u	2u	3u	1r	2r	1t	2t

Cut Scores

Not Achieved	Achievement	Achievement with Merit	Achievement with Excellence
0 – 7	8– 12	13 – 18	19 – 24

(e) A curve is defined by the equation $y = \frac{2x^2 - 1 - 2x\ln x}{x}$, where $x > 0$.

The curve has a point of inflection at the point P.

Find the equation of the tangent to the curve at the point P.

You must use calculus and show any derivatives that you need to find when solving this problem.

(e)	$\frac{dy}{dx} = 2 + x^{-2} - \frac{2}{x}$ $\frac{d^2y}{dx^2} = -2x^{-3} + 2x^{-2}$ For an inflection, solve $\frac{d^2y}{dx^2} = -2x^{-3} + 2x^{-2} = 0$ $\frac{2}{x^2} = \frac{2}{x^3}$ $2x^3 = 2x^2$ $2x^3 - 2x^2 = 0$ $2x^2(x-1) = 0$ Either $x = 0$ ignore as not valid Or $x = 1$ $x = 1$ gives $y = 1$, i.e. $P = (1,1)$ $x = 1$ gives $\frac{dy}{dx} = 1$ Then equation of tangent is: $y - 1 = 1(x-1)$ $y = x$	• Correct expression for $\frac{dy}{dx}$.	• Finds $x = 1$, with correct $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.	• E7 / T1 Consistent equation of tangent, from incorrect P. OR Correct solution but with one minor error. • E8 / T2 Correct equation of tangent found.

NØ	N1	N2	A3	A4	M5	M6	E7	E8
No response; no relevant evidence.	ONE partial solution.	1u	2u	3u	1r	2r	T1	T2

(e) The graph of the function $y = \dfrac{xe^{3x}}{2x+k}$, where k is a non-zero constant, **has a single turning point** at **Q**.

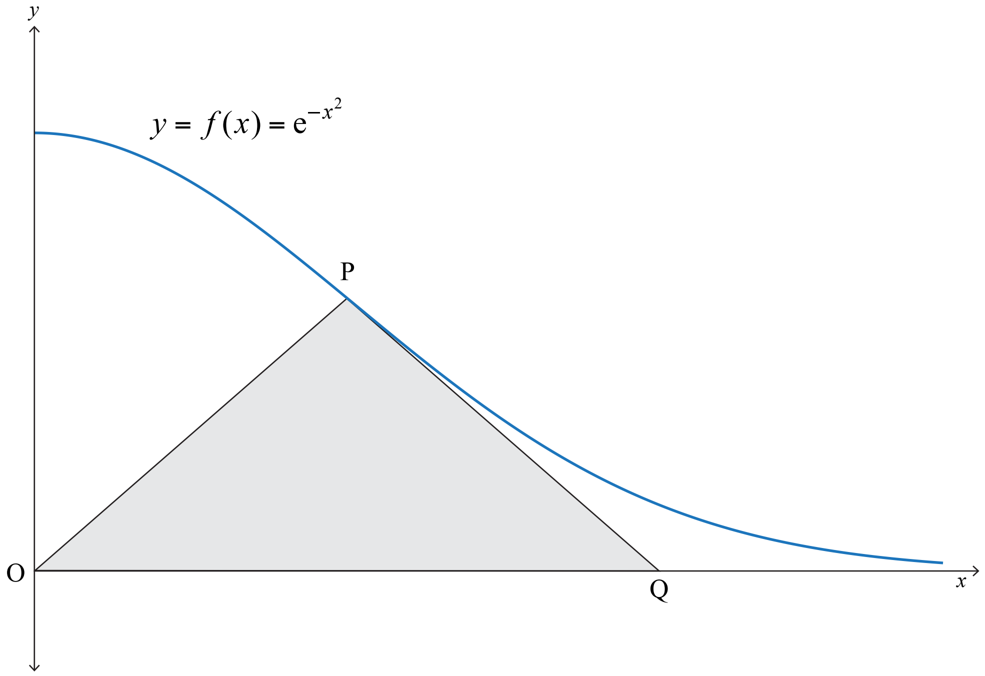
Find the x-coordinate of the point Q.

You must use calculus and show any derivatives that you need to find when solving this problem.

(e)	$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{e}^{3x}(6x^2+3kx+k)}{(2x+k)^2}$ For turning points $\frac{\mathrm{d}y}{\mathrm{d}x}=0$ $\mathrm{e}^{3x}(6x^2+3kx+k)=0$ Either $e^{3x}=0$ No solutions Or $6x^2+3kx+k=0$ (a) But, as only a single turning point, using $b^2-4ac=0$ $(3k)^2-4\times6\times k=0$ $9k^2-24k=0$ $3k(3k-8)=0$ Either $k=0$ Ignore as not valid Or $3k-8=0$ i.e. $k=\frac{8}{3}$ Substituting $k=\frac{8}{3}$ into equation (a) gives $6x^2+8x+\frac{8}{3}=0$ $9x^2+12x+4=0$ $(3x+2)(3x+2)=0$ $x=-\frac{2}{3}$	• Correct $\frac{\mathrm{d}y}{\mathrm{d}x}$.	• Finds the correct value of k, with evidence of calculus methods.	• E7 / T1 Correct solution but with one minor error. • E8 / T2 Calculates x-co-ordinate of Q, with clear and full calculus justification.

NØ	N1	N2	A3	A4	M5	M6	E7	E8
No response; no relevant evidence.	ONE partial solution.	1u	2u	3u	1r	2r	T1	T2

(e) The diagram below shows part of the graph of the function $f(x) = \mathrm{e}^{-x^2}$, where $x \geq 0$.



The point P lies on the curve and the point Q lies on the x-axis so that OP = PQ, where O is the origin.

Prove that the largest possible area of the triangle OPQ is $\dfrac{1}{\sqrt{2\mathrm{e}}}$.

You do not need to show that the area you have found is a maximum.

You must use calculus and show any derivatives that you need to find when solving this problem.

(e)	Let P have coordinates (x, y) Area of triangle $OPQ = xe^{-x^2}$ $\frac{dA}{dx} = e^{-x^2}(1-2x^2)$ For max / min, $\frac{dA}{dx} = 0$ $e^{-x^2}(1-2x^2) = 0$ Either $e^{-x^2} = 0$ No solutions Or $(1-2x^2) = 0$ i.e. $x = \pm\frac{1}{\sqrt{2}} = \pm 0.7071$ But ignore $x = -\frac{1}{\sqrt{2}}$ as $x > 0$ Then area $= \frac{1}{\sqrt{2}} \times e^{-\frac{1}{2}} = \frac{1}{\sqrt{2}} \times \frac{1}{e^{\frac{1}{2}}}$ Area $= \frac{1}{\sqrt{2e}}$ as required.	• Correct expression for $\frac{dA}{dx}$.	• Finding $x = \frac{1}{\sqrt{2}}$, and evidence of ignoring $x = -\frac{1}{\sqrt{2}}$ with evidence of a calculus method.	• E7 / T1 Correct proof, but with one minor error. • E8 / T2 Correct proof of exact area value, with evidence of a calculus method.

NØ	N1	N2	A3	A4	M5	M6	E7	E8
No response; no relevant evidence.	ONE partial solution.	1u	2u	3u	1r	2r	T1	T2

Cut Scores

Not Achieved	Achievement	Achievement with Merit	Achievement with Excellence
0 – 07	08– 12	13 – 18	19– 24